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A strain gradient linear viscoelasticity theory

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ABSTRACT

In this paper, a strain gradient viscoelastic theory is proposed strictly, which can be used to describe the cross-scale mechanical behavior of the quasi-brittle advanced materials. We also expect the theory to be applied to the description for the cross-scale mechanical behavior of advanced alloy metals in linear elastic deformation cases. In the micro-/nano-scale, the mechanical properties of advanced materials often show the competitive characteristics of strengthening and softening, such as: the strength and hardness of the thermal barrier coatings with nanoparticles and the nanostructured biological materials (shells), as well as the strength of nanocrystalline alloy metals which show the characteristics of positive-inverse Hall-Petch relationship, etc. In order to characterize these properties, a strain gradient viscoelastic theory is established by strictly deriving the correspondence principle. Through theoretical derivation, the equilibrium equations and complete boundary conditions based on stress and displacement are determined, and the correspondence principle of strain gradient viscoelasticity theory in Laplace phase space is obtained. With the help of the high-order viscoelastic model, the specific form of viscoelastic parameters is presented, and the time curve of material characteristic scale parameters in viscoelastic deformation is obtained. When viscoelasticity is neglected, the strain gradient viscoelasticity theory can be simplified to the classical strain gradient elasticity theory. When the strain gradient effect is neglected, it can be simplified to the classical viscoelastic theory. As an application example of strain gradient viscoelastic theory, the solution to the problem of cross-scale viscoelastic bending of the Bernoulli-Euler beam, is analyzed and presented.

1. Introduction

Generally speaking, considering the viscous effect of material and strictly solving its mechanics response, it will be faced with solving the solution of an initial boundary value for a three-dimensional dynamic problem, which usually requires the Laplace transform and inverse Laplace transformation, which is an extremely difficult mechanics solution process. Although many research works have been carried out in this area in the last several decades, the substantial research progress is only limited to the characterization of linear viscoelasticity of materials. In this respect, the corresponding principle for solving the linear viscoelastic effect is strictly established. As long as the solution of the linear elastic problem is obtained, the corresponding solution of the linear viscoelastic problem can be obtained strictly through using the inverse Laplace transformation based on the correspondence principle. According to the authors' understanding, up to now, except for the linear viscoelastic case, there has not existed the correspondence principle for any other viscous cases, although some approximate models and the approximate analysis methods are developed for other viscoelastic and viscoplastic cases in last several decades.

In the present research, considering that it is essential to describe the cross-scale mechanical behavior for a wide class of advanced elastic-brittle materials, such as, the advanced thermal barrier coatings with nanoparticles (Liu et al., 2018), nanostructured biological materials (such as shells) (Song et al., 2016), amorphous glass (Cui et al., 2017), etc., we intend to present a strain gradient linear viscoelastic theory through developing the correspondence principle. We expect that the strain gradient linear viscoelastic theory can describe not only the cross-scale mechanical behavior of the advanced elastic-brittle materials, but also some alloy metals in elastic deformation cases when the applied load is not large.

At the micro-/nano-scale, the mechanical properties of advanced materials often show the competitive characteristics of strengthening and softening. Strengthening corresponds to the strain gradient effect while softening corresponds to the viscous effect. In the past decades, a lot of researches has been carried

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out on the strengthening and softening properties of advanced materials. Micro-/nano-structured materials are widely used in different fields, such as aviation and aerospace, medicine, and engineering (Liu et al., 2018; Cui et al., 2017; Moon et al., 2011; Koumoulos et al., 2015; Zhong and Yan, 2016; Morales-Rivas et al., 2015; Lurie et al., 2003). A large number of experimental observations showed that at the micro-/nano-scale, the mechanical behavior of materials shows a strong scale dependence (Ma and Clarke, 1995; Lam et al., 2003; Mcfarland and Colton, 2005; Lei et al., 2016). Furthermore, the softening effect of viscosity in experiments was also observed. Long et al. (Long et al., 2018) found that Young's modulus and hardness are strain rate dependent, and they discussed the creep behavior during the holding stage. The strain rate sensitivity of pure aluminum in nanoindentation experiments were also presented by Yamada et al. (Yamada et al., 2013). Li et al. (Li et al., 2010) conducted a large-scale molecular dynamics simulation study on nano-twin copper, and the results showed that the material strength softened below the critical twin thickness, showing the inverse Hall-Petch effect. Jun et al. (Jun et al., 2011) performed the bulge test of graphene monolayers with molecular dynamics simulations and found the size-dependent nonlinear elastic softening of graphene. Li et al. (Li et al., 2005) completed the nanoindentation experiment on ZnS nanobelts at different temperatures. They discovered that ZnS nanobelts exhibit significant creep behavior under constant load not only at high temperatures but also at room temperature. Therefore, there is an urgent need for a new deformation mechanism to consider both the strain gradient effect and the viscous effect.

In the past few decades, the scale-dependent deformation behavior of materials has attracted extensive attention. For example, the elastic modulus of polymer epoxy a function of beam thickness in the bending and tension experiments (Lam et al., 2003), the size effect is significant when the beam thickness is comparable to the material length scale parameter in beam bending tests (Mcfarland and Colton, 2005), the torsion experiment of the copper wire with the diameter of micrometer shows that the torsion shear strength changes dramatically with the diameter of the copper wire (Fleck et al., 1994), and so on. These mentioned above cross-scale mechanical behavior can be described by the theory of higher-order continuum mechanics, i.e., strain gradient theory. In the past few decades, different versions of strain gradient theory have been proposed. Mindlin proposed a generalized strain gradient theory (Mindlin, 1964), in which the classical strain and strain gradient constitute the measure of total deformation, including 16 additional material parameters in addition to lame constant. Mindlin and Eshel reduced the length scale parameters of isotropic materials from 16 to 5 (Mindlin and Eshel, 1968) in the second-order strain gradient theory. However, compared with the classical continuum mechanics theory, five additional material parameters are still difficult to be determined by experiments. In view of this, based on the strain gradient theory proposed by Mindlin, Aifantis et al. put forward a simplified theory (Altan and Aifantis, 1997; Aifantis, 1992) with only one scale parameter, which can be measured experimentally. This theoretical model is regarded as a special case of Mindlin's theory (Askes and Aifantis, 2011; Lazar and Maugin, 2005). Then, by using the principle of minimum potential energy, a variational formula of the simplified strain gradient elasticity theory is given, and a complete boundary condition is derived. Because of its simple form, the theoretical model is applied to analyze different problems, such as bending problems (Papargyri-Beskou et al., 2003; Lurie and Solyaev, 2018; Lazopoulos, 2009), torsion problems (Liu, 2013; Lazopoulos and Lazopoulos, 2012) and other problems (Sidhardh and Ray, 2018; Mühlich et al., 2012; Gao and Park, 2007). For the theory of strain gradient plasticity, the scale effect of plastic deformation is mainly concerned. Fleck and Hutchinson (Fleck and Hutchinson, 1993, 1997) extended Mindlin's simplified strain gradient elasticity theory to the plastic case and developed the plastic strain gradient theory. Wei and Hutchinson (Wei and Hutchinson, 1997) further developed Fleck and Hutchinson's theories by emphasizing the compressibility of materials and the contribution of the elastic strain gradient. Fleck and Hutchinson's strain gradient theory have been applied to the characterization of multiple effects (Qu et al., 2006; Huang et al., 2004; Wei and Hutchinson, 2003; Wei, 2006; Donà et al., 2014; Wei et al., 2001).

For the micro-/nano-scale mechanical properties experiment of soft materials, the size effect and viscous effect of mechanical behavior are also observed, such as polymer (Chong and Lam, 1999; Voyiadjis and Deliktas, 2009; Nikolov et al., 2007), biomaterial (Philippart et al., 2015; Honglin, 2017; Oyen, 2013), epoxy resin (Lam and Chong, 2000; Alisafaei et al., 2014), etc. Obviously, the viscous effect of soft materials will be more prominent. In this case, a new high-order measurement method has been developed to characterize the strain gradient behavior and to explain the size effect of mechanical behavior of polymer epoxy beams when bending (Lam et al., 2003). Nikolov et al. (Nikolov et al., 2007) put forward the theory of strain gradient elasticity of solid polymer based on the concept of Frank's elasticity, in which the size effect is only related to the rotational strain gradient. The theoretical calculation shows that the length scale of rubber and liquid crystal polymer in the case of small strain elasticity is about several nanometers. Han et al. (Han and Nikolov, 2007) extended Nikolov et al.'s strain gradient elasticity theory to the elastoplastic case. Similar to the mechanism-based strain gradient plasticity theory (Gao et al., 1999), some gradient-dependent plasticity models have been established to explain the mechanical behavior of polymer under the size dependence (Chong and Lam, 1999; Voyiadjis and Deliktas, 2009; Swaddiwudhipong et al., 2005). In these models, the concept of geometrically necessary buckling (Argon, 1973) is introduced to explain the scale effect in the plastic deformation process. Soft materials are widely used in the field of medical engineering, which also show significant viscous effect at the micro-/nano-scale (Crichton et al., 2013; Chen et al., 2014; Li, 2018; Tranchida et al., 2009).

The micro-/nano-scale viscous effect and strain gradient effect are obvious. Thus, some theories have been developed to explain these phenomena. Valanis (Valanis, 1997) proposed the gradient theory of viscoelasticity by using the variational principle and the notion of internal fields under isothermal conditions. Iesan and Quintanilla (Ieşan and Quintanilla, 2013) developed a gradient theory of thermoviscoelasticity and presented a thermoviscoelastic solution that is analogous to the Cauchy–Kowalewski–Somigliana classical elastic solution in the isothermal theory. Based on the strain gradient plasticity theory proposed by Fleck and Hutchinson (Fleck and Hutchinson, 2001) and the finite strain gradient plasticity by Niordson and Redanz (Niordson, 2004), Borg et al. (Borg et al., 2006) proposed a viscoplastic generalization. Also, a finite element framework was finished to analyze materials with voids or inclusions. Lele and Anand (Lele and Anand, 2009) developed a thermodynamically consistent large-deformation theory of strain-gradient viscoplasticity and solved some simple mechanical problems using this theory. However, these studies did not include the high-order viscosity or derive the formulation of gradient parameters.

In this study, we propose a strain gradient linear viscoelasticity theory, which can describe the viscous effect and strain gradient effect of solid at micro-/nano-scale. At the same time, the expressions of gradient parameters related to the viscous effect and strain gradient effect are derived in detail by building a high-order viscoelastic model. The rest of this paper is arranged as follows. In the second section, the simplified strain gradient elasticity theory and the classical viscoelastic theory are briefly introduced. In the

third section, based on the variational principle, the strain gradient viscoelastic theory is established, and the equilibrium equation and boundary conditions are given. Then, a correspondence principle between the strain gradient viscoelastic theory (in the Laplace phase space) and the strain gradient elastic theory is proposed. In the fourth section, the relationship between the gradient parameters and viscoelasticity is obtained by establishing a high-order viscoelastic model. The fifth section takes the bending of beam as an example. The last section 6 is the conclusion.

2. Related theories

In this section, we give a briefly reviewing on the related theories, the simplified strain gradient elasticity theory developed by Altan and Aifantis (Altan and Aifantis, 1997)and Gao and Park (Gao and Park, 2007), and the classical linear viscoelastic theory.

2.1. The simplified strain gradient elasticity theory

Based on Mindlin's general strain gradient, a simplified strain gradient elasticity theory was proposed by Altan and Aifantis (Altan and Aifantis, 1997; Aifantis, 1992). Then, Gao and Park (Gao and Park, 2007) provided a variational formulation by using the principle of minimum total potential energy. This simplified theory only involves an extra gradient parameter besides the Lame constants.

For isotropic and elastic material, the strain energy density considering strain gradient effect is written as (Altan and Aifantis, 1997; Gao and Park, 2007)

$$w = w(\varepsilon_{ij}, \varepsilon_{ijk}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + G\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}A_1\varepsilon_{iik}\varepsilon_{jjk} + A_2\varepsilon_{ijk}\varepsilon_{ijk} \tag{1}$$

where λ and G are Lame Constants, A_1 and A_2 denote the high order moduli, the strain ε_{ij}, and strain gradient ε_{ijk} are related to displacement, given as

$$\begin{aligned} \varepsilon_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ji} \\ \varepsilon_{ijk} &= \varepsilon_{ij,k} = \tfrac{1}{2}(u_{i,jk} + u_{j,ik}) = \varepsilon_{jik} \end{aligned} \tag{2}$$

The high-order moduli are represented by using a gradient parameter c_e and Lame constants in the simplified strain gradient elasticity theory, as

$$A_1 = c_e\lambda, A_2 = c_eG \tag{3}$$

where c_e has the dimension of length squared.

Referring to Eq. (3), Eq. (1) is a simplified formulation of the gradient elasticity of Mindlin (Mindlin, 1964), proposed by Altan and Aifantis (Altan and Aifantis, 1997).

The total strain energy for gradient-based linear elastic material can be written as

$$W = \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} + \tau_{ijk}\varepsilon_{ijk})\mathrm{d}V \tag{4}$$

where the components of the Cauchy stress σ_{ij}, the double stress τ_{ijk} are given as follows

$$\begin{aligned} \sigma_{ij} &= \frac{\partial w}{\partial \varepsilon_{ij}} = \delta_{ij}\lambda\varepsilon_{pp} + 2G\varepsilon_{ij} = \sigma_{ji}\tau_{ijk} = \frac{\partial w}{\partial \varepsilon_{ijk}} \\ &= c_e(\delta_{ij}\lambda\varepsilon_{ppk} + 2G\varepsilon_{ijk}) = \tau_{jik} \end{aligned} \tag{5}$$

The external force potential energy is

$$P = \int_V f_i u_i \mathrm{d}V + \int_S (p_i u_i + q_i D u_i)\mathrm{d}S \tag{6}$$

where f_i, p_i and q_i are body forces, surface tractions, and double tractions, respectively. D is the normal derivative operator

$$D \equiv n_i \frac{\partial}{\partial x_i} \tag{7}$$

with the outward unit normal vector $\mathbf{n} = n_i\mathbf{e}_i$. Using the principle of minimum total potential energy, the variation of total potential energy equals to zero:

$$\begin{aligned} \delta\Pi &= \delta W - \delta P \\ &= \int_V (\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\varepsilon_{ijk} - f_i\delta u_i)\mathrm{d}V - \int_S (p_i\delta u_i + q_i D(\delta u_i))\mathrm{d}S \end{aligned} \tag{8}$$

After some calculations, one can obtain the following governing equilibrium equations and boundary conditions as,

$$(\sigma_{ij} - \tau_{ijk,k})_{,j} + f_i = 0 \tag{9}$$

$$\begin{aligned} &(\sigma_{ij} - \tau_{ijk,k})n_j - D_j(n_k\tau_{ijk}) + (D_l n_l)n_k n_j \tau_{ijk} = \bar{p}_i \text{ or } u_i \\ &= \bar{u}_i \tau_{ijk} n_j n_k = \bar{q}_i \text{ or } u_{i,l}n_l = \frac{\partial \bar{u}_i}{\partial n} \end{aligned} \tag{10}$$

whereD_j is the surface gradient operator

$$D_j = \frac{\partial}{\partial x_j} - n_j D \tag{11}$$

Eqs. (2), (5), (9) and (10) compose the simplified strain gradient elasticity theory, and only one material parameter is involved in this theory.

2.2. Viscoelastic theory

Viscoelastic theories have been developed in the past hundred years. Viscoelastic constitutive models are useful to model phenomena such as creep, relaxation, damping etc. In this section, we give briefly reviewing on the viscoelastic theory, mainly refer to Christensen (Christensen, 1982).

In viscoelastic models, the current stress depends on the strain history. If the strain history $\varepsilon_{ij} = \varepsilon_{ij}(t)$ are assumed to be continuous, and $\varepsilon_{ij}(t) = 0$ for $t < 0$, the stress–strain relation can be expressed using a Stieltjes integral

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-\zeta)\dot{\varepsilon}_{kl}(\zeta)\mathrm{d}\zeta = \int_{-\infty}^{t} C_{ijkl}(t-\zeta)\mathrm{d}\varepsilon_{kl}(\zeta) \tag{12}$$

where dot above the variable denotes derivatives, the integrating stiffness function $C_{ijkl}(t)$ is a fourth-order tensor.

For concisely, the stress-strain relation of Eq. (12) is used to be written as the Stieltjes convolution form

$$\sigma_{ij}(t) = C_{ijkl}(t) * \mathrm{d}\varepsilon_{kl}(t) = \varepsilon_{kl}(t) * \mathrm{d}C_{ijkl}(t) \tag{13}$$

where $*$ is the Stieltjes convolution symbol. The Stieltjes convolution is defined as

$$\phi * \mathrm{d}\varphi = \int_{-\infty}^{t} \phi(t-\zeta)\dot{\varphi}(t)\mathrm{d}t \tag{14}$$

Similarly, the isotropic form of the viscoelastic stress-strain relations can be expressed by the spherical tensor and deviatoric tensor of stress and strain

$$s_{ij}(t) = 2G(t) * \mathrm{d}e_{ij}(t)\sigma_{pp}(t) = 3K(t) * \mathrm{d}\varepsilon_{pp}(t) \tag{15}$$

where $G(t)$ is the shear relaxation function and $K(t)$ is the bulk relaxation function. Total stress tensor is given as

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\delta_{ij}\sigma_{pp} = 2G * \mathrm{d}\varepsilon_{ij} + \delta_{ij}\lambda * \mathrm{d}\varepsilon_{pp} \tag{16}$$

where $G(t)$ and $\lambda(t)$, similar to the Lame constants in elasticity, are the relaxation functions in viscoelasticity. The strain at each moment is related to the displacement at that moment, given by

$$\varepsilon_{ij} = \varepsilon_{ij}(t) = \frac{1}{2}\left[u_{i,j}(t) + u_{j,i}(t)\right] \tag{17}$$

In order to solve viscoelastic problems, Laplace transformation is often used. the Laplace transformation of Eqs. and are given as follows, noting the condition $\varepsilon_{ij} = 0$ for $t < 0$,

$$\bar{s}_{ij} = 2s\bar{G}\bar{e}_{ij}\bar{\sigma}_{pp} = 3s\bar{K}\bar{\varepsilon}_{pp}\bar{\sigma}_{ij} = 2s\bar{G}\bar{\varepsilon}_{ij} + \delta_{ij}s\bar{\lambda}\bar{\varepsilon}_{pp} \tag{18}$$

where s is the Laplace transformation variable, symbols with a bar above denote variables undergone the Laplace transformation. It can be found that the elasticity relationship is reinterpreted as the Laplace transform of the corresponding viscoelastic relationship by replacing G by $s\bar{G}(s)$, λ by $s\bar{\lambda}(s)$, and K by $s\bar{K}(s)$. This is the correspondence principle between elasticity and viscoelasticity. The viscoelastic problems can be solved directly based on the elastic solutions by using the correspondence principle.

However, the form of relaxation functions should be specific to obtain the solutions of viscoelastic problems. There are several different models to explain the viscoelastic property of materials. Here, the most commonly used standard three-parameter model is presented as an example, as shown in Fig. 1. The classical standard three-parameter model is always referred to as the model of linear solid materials. The series combination of a spring and a dashpot is an important building block, which called Maxwell unit. This three-parameter model is obtained by adding a spring in parallel to a Maxwell unit. For isotropic materials, the stress–strain relation can be written as

$$\sigma_e = \sigma_0 + \sigma_1\dot{\varepsilon}_e = \frac{\dot{\sigma}_1}{E_1} + \frac{\sigma_1}{\eta_1}\varepsilon_e = \frac{\sigma_0}{E_0} \tag{19}$$

where σ_e and ε_e are equivalent stress and strain of the model, respectively. σ_0 and σ_1 are the stress acting on the spring and the Maxwell unit, respectively.

Applying the Laplace transformation to Eq. (19)

$$\bar{\sigma} = \left(E_0 + \frac{E_1\tau_g s}{1+\tau_g s}\right)\bar{\varepsilon} = Q(s)\,\bar{\varepsilon} \tag{20}$$

where $\tau_g = \eta_1/E_1$ is the relaxation time of the Maxwell unit in macroscopic scale. The relaxant $Q(s)$ is defined as

$$Q(s) = E_0 + \frac{E_1\tau_g s}{1+\tau_g s} \tag{21}$$

The relaxation function in the Laplace space is defined as

$$\bar{E}(s) = \frac{Q(s)}{s} = \frac{E_0}{s} + \frac{E_1\tau_g}{1+\tau_g s} \tag{22}$$

Applying the inverse Laplace transformation, the relaxation modulus in Euler space can be obtained

$$E(t) = L^{-1}E(s) = E_0 + E_1 e^{-t/\tau_g} \tag{23}$$

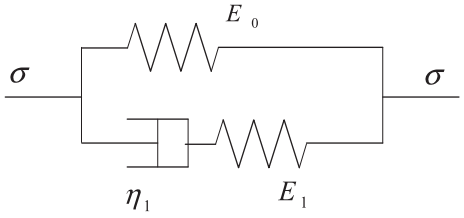

Fig. 1. The classical three-parameter viscoelastic model.

There are many different viscoelastic mechanical models, such as standard series–parallel models (a spring in series with a Voigt unit or parallel with a Maxwell unit), and other different combinations between springs and dashpots. They are applied in different problems to explain different viscous behaviors.

The simplified function of the first viscoelastic variational principle, $\prod$, is given as (Christensen, 1982), provided that the accumulation strain is taken as the independent variables of the strain energy density function,

$$\Pi = \int_V \left[\frac{1}{2}C_{ijkl} * \mathrm{d}\varepsilon_{ij} * \mathrm{d}\varepsilon_{kl} - f_i * \mathrm{d}u_i\right]\mathrm{d}V - \int_S (p_i * \mathrm{d}u_i)\mathrm{d}S \tag{24}$$

where $C_{ijkl}(t)$ is the stiffness function, f_i are body forces, p_i are the prescribed stresses on the boundary. The first variation of Eq. (25) is given by

$$\delta\Pi = \int_V \left[C_{ijkl} * \mathrm{d}\varepsilon_{ij} * \mathrm{d}(\delta\varepsilon_{kl}) - f_i * \mathrm{d}(\delta u_i)\right]\mathrm{d}V - \int_S (p_i * \mathrm{d}(\delta u_i))\mathrm{d}S \tag{25}$$

The variational equation $\delta\prod = 0$ leads to the equilibrium equations and the stress boundary condition (Christensen, 1982)

$$\sigma_{ij,j} + f_i = 0 \tag{26}$$

$$\sigma_{ij}n_j = p_i \tag{27}$$

This simplified viscoelastic variational principle is an extension of the principle of potential energy in elasticity, and it is a simplified form of Gurtin's (Gurtin, 1963) linear viscoelastic variational theorems.

3. Strain gradient viscoelasticity theory

In this section, a strain gradient viscoelasticity theory is generated by extending and combining the strain gradient elasticity and conventional viscoelasticity. In the strain gradient viscoelasticity, components of stress, double stress, strain, strain gradient are functions of time, besides of the coordinates, and equations are described by the Stieltjes convolution.

3.1. Variational formulation for strain gradient linear viscoelastic

In conventional viscoelasticity, the double stress and strain gradient are neglected. The stress–strain relation is given as Eq. (13), and the strain is related to displacement at the current moment, as Eq. (17). However, the size effect should also be significant at the micro-/nano-scale. Thus, in our gradient viscoelasticity, size effect should be considered. The strain gradient is defined as

$$\varepsilon_{ijk}(t) = \varepsilon_{ij,k}(t) = \frac{1}{2}\left[u_{i,jk}(t) + u_{j,ik}(t)\right] = \varepsilon_{jik}(t) \tag{28}$$

And $\varepsilon_{ijk}(t) = 0$ for $t < 0$. The work-conjugate double stress τ_{ijk} is related to the strain gradient rate and deformation history. Thus, the relation between double stress and strain gradient can be written as the Stieltjes convolution

$$\tau_{ijk}(t) = A_{ijklnm}(t) * \mathrm{d}\varepsilon_{lmn}(t) \tag{29}$$

In Eq. (29), we introduce a new material parameter $A_{ijklmn}(t)$, which is called high-order relaxation function. It is a sixth-order tensor.

Considering an assumption of point symmetry in the symmetry group of the material, and by analogy with Eq. (8), provided that the accumulation strain and accumulation strain gradient are taken as independent variables of the total strain energy density

function, the variational function $\prod$ for the strain gradient linear viscoelastic theory can be derived as follows,

$$\Pi = \frac{1}{2}\int_V (C_{ijkl} * \mathrm{d}\varepsilon_{ij} * \mathrm{d}\varepsilon_{kl} + A_{ijklmn} * \mathrm{d}\varepsilon_{ijk} * \mathrm{d}\varepsilon_{lmn})\mathrm{d}V - \int_V (f_i * \mathrm{d}u_i)\mathrm{d}V - \int_S (p_i * \mathrm{d}u_i + q_i * D\mathrm{d}u_i)\mathrm{d}S \tag{30}$$

where $f_i(t)$ are body forces, $p_i(t)$ and $q_i(t)$ denote the stress traction and double stress traction on the boundary, respectively. The normal derivative operator $D(\cdot)$ is defined as Eq. (7).

Using Eqs. (13) and (29), the function $\prod$ can be described by the stress and double stress

$$\Pi = \frac{1}{2}\int_V (\sigma_{ij} * \mathrm{d}\varepsilon_{ij} + \tau_{ijk} * \mathrm{d}\varepsilon_{ijk})\mathrm{d}V - \int_V (f_i * \mathrm{d}u_i)\mathrm{d}V - \int_S (p_i * \mathrm{d}u_i + q_i * D\mathrm{d}u_i)\mathrm{d}S \tag{31}$$

The resulting first variation of Eq. (31) is given by

$$\delta\Pi = \int_V [\sigma_{ij} * \mathrm{d}(\delta\varepsilon_{ij}) + \tau_{ijk} * \mathrm{d}(\delta\varepsilon_{ijk})]\mathrm{d}V - \int_V f_i * \mathrm{d}(\delta u_i)\mathrm{d}V - \int_S \{p_i * \mathrm{d}(\delta u_i) - q_i * \mathrm{d}[D(\delta u_i)]\}\mathrm{d}S \tag{32}$$

Using Eqs. (17) and (28), and applying the chain rule, the first term on the right-hand side of Eq. (32) can be written as

$$\begin{aligned}&\int_V [\sigma_{ij} * \mathrm{d}(\delta\varepsilon_{ij}) + \tau_{ijk} * \mathrm{d}(\delta\varepsilon_{ijk})]\mathrm{d}V\\ &= \int_V [\sigma_{ij} * \mathrm{d}(\delta u_{i,j}) + \tau_{ijk} * \mathrm{d}(\delta u_{i,jk})]\mathrm{d}V\\ &= \int_V \{[(\sigma_{ij} - \tau_{ijk,k}) * \mathrm{d}(\delta u_i)]_{,j} - [(\sigma_{ij} - \tau_{ijk,k})]_{,j} * \mathrm{d}(\delta u_i)\\ &\quad + [\tau_{ijk} * \mathrm{d}(\delta u_{i,j})]_{,k}\}\mathrm{d}V\end{aligned} \tag{33}$$

Further, applying divergence theorem to Eq. (33) results in

$$\begin{aligned}&\int_V [\sigma_{ij} * \mathrm{d}(\delta\varepsilon_{ij}) + \tau_{ijk} * \mathrm{d}(\delta\varepsilon_{ijk})]\mathrm{d}V\\ &= -\int_V (\sigma_{ij} - \tau_{ijk,k})_{,j} * \mathrm{d}(\delta u_i)\mathrm{d}V + \int_S n_j(\sigma_{ij} - \tau_{ijk,k}) * d(\delta u_i)\mathrm{d}S + \int_S n_k\tau_{ijk} * \mathrm{d}(\delta u_{i,j})\mathrm{d}S\end{aligned} \tag{34}$$

The integrand of the last integral of Eq. (34) can be written as

$$n_k\tau_{ijk} * \mathrm{d}(\delta u_{i,j}) = n_k\tau_{ijk} * \mathrm{d}D_j(\delta u_i) + n_k n_j\tau_{ijk} * \mathrm{d}D(\delta u_i) \tag{35}$$

where D_j has been defined in Eq. (11). Using the variational chain rule, the first term on the right-hand side of Eq. (35) can be written as

$$n_k\tau_{ijk} * \mathrm{d}D_j(\delta u_i) = D_j[n_k\tau_{ijk} * \mathrm{d}(\delta u_i)] - D_j(n_k\tau_{ijk}) * \mathrm{d}(\delta u_i) \tag{36}$$

And

$$D_j[n_k\tau_{ijk} * \mathrm{d}(\delta u_i)] = (D_l n_l)n_k n_j\tau_{ijk} * \mathrm{d}(\delta u_i) + n_q e_{qpm}(e_{mlj}n_l n_k\tau_{ijk} * \mathrm{d}(\delta u_i))_{,p} \tag{37}$$

where e_{ijk} is the permutation symbol. Similar results have been given by Mindlin (Mindlin, 1964) and proved by Gao and Park (Gao and Park, 2007) in the derivation of strain gradient elasticity theory. However, the integral of time replaces the matrix multiplication in our strain gradient viscoelasticity theory.

Combining Eqs. (34)–(37) then gives

$$\begin{aligned}&\int_V (\sigma_{ij} * \mathrm{d}\delta\varepsilon_{ij} + \tau_{ijk} * \mathrm{d}(\delta\varepsilon_{ijk}))\mathrm{d}V\\ &= -\int_V (\sigma_{ij} - \tau_{ijk,k})_{,j} * \mathrm{d}(\delta u_i)\mathrm{d}V + \int_S n_k n_j\tau_{ijk} * \mathrm{d}(D\delta u_i)\mathrm{d}S\\ &\quad + \int_S [n_j(\sigma_{ij} - \tau_{ijk,k}) - D_j(n_j\tau_{ijk}) - (D_l n_l)n_k n_j\tau_{ijk}]\\ &\quad * d(\delta u_i)\mathrm{d}S + \int_S n_q e_{pqm}(e_{mlj}n_l n_k\tau_{ijk} * \mathrm{d}(\delta u_i))_{,p}\mathrm{d}S\end{aligned} \tag{38}$$

Assuming the boundary S is smooth, the last integral term of Eq. (38) is zero according to Stokes' theorem. Hence, the first variation of Eq. (31) can be written as

$$\begin{aligned}\delta\Pi = &-\int_V [(\sigma_{ij} - \tau_{ijk,k})_{,j} + f_i] * \mathrm{d}(\delta u_i)\mathrm{d}V\\ &+ \int_S [(\sigma_{ij} - \tau_{ijk,k})n_j - D_j(n_k\tau_{ijk}) + (D_l n_l)n_k n_j\tau_{ijk} - p_i]\\ &* d(\delta u_k)\mathrm{d}S + \int_s (\tau_{ijk}n_j n_k - q_i) * \mathrm{d}D(\delta u_i)\mathrm{d}S\end{aligned} \tag{39}$$

Let $\delta\Pi = 0$, the differential equilibrium equation can be obtained

$$(\sigma_{ij} - \tau_{ijk,k})_{,j} + f_i = 0 \tag{40}$$

together with the boundary conditions

$$(\sigma_{ij} - \tau_{ijk,k})n_j - D_j(n_k\tau_{ijk}) + (D_l n_l)n_k n_j\tau_{ijk} = p_i \text{ or } u_i = \hat{u}_i \tag{41}$$

$$\tau_{ijk}n_j n_k = q_i \text{ or } u_{i,l}n_l = \frac{\partial\hat{u}_i}{\partial n} \tag{42}$$

The format of the governing equilibrium equation and the boundary conditions derived here are similar to those given by Altan and Aifantis (Altan and Aifantis, 1997) and Gao and Park (Gao and Park, 2007). However, all variables in Eqs. (40)-(42) are the function of time. Namely, they are related to the history of deformation. That is why our strain gradient viscoelasticity theory can explain the time-dependent property of materials. In addition, the high-order boundary condition of displacement occurs due to the introduction of high-order deformations.

If the viscosity is not considered, variables will be time-independent, and there is no need to describe equations in the Stieltjes convolution. Hence, the equilibrium equation and boundary conditions reduce to the simplified strain gradient elasticity theory given by Gao and Park, as Eqs. (9) and (10). Furthermore, in the absence of the size effect, the equations - reduce to,

$$\sigma_{ij,j} + f_i = 0\ \sigma_{ij}n_j = p_i \text{ or } u_i = \hat{u}_i \tag{43}$$

where σ_{ij} becomes the components of Cauchy stress in classical elasticity. Eq. (43) explains the governing equation and boundary conditions in the classical elastic theory.

3.2. Gradient viscoelasticity for linear isotropic solids

For gradient-dependent, isotropic, elastic materials, stress is written as Eq. (16). Similarly, the double stress can be expressed in the spherical and deviatoric of the strain gradient

$$\tau_{ijk} = \frac{\partial w}{\partial\varepsilon_{ijk}} = \delta_{ij}A_1 * \mathrm{d}\varepsilon_{ppk} + 2A_2 * \mathrm{d}\varepsilon_{ijk} \tag{44}$$

where $A_1(t)$ and $A_2(t)$ are higher-order relaxation functions for isotropic solids, which are similar to the definition of Lame constants in classical elasticity.

Combining Eqs. (16), (40), and (44), the equilibrium equation leads to

$$\left[\left(\delta_{ij}\lambda * \mathrm{d}\varepsilon_{pp} + 2G * \mathrm{d}\varepsilon_{ij}\right) - \left(\delta_{ij}A_1 * \mathrm{d}\varepsilon_{ppk} + 2A_2 * \mathrm{d}\varepsilon_{ijk}\right)_{,k}\right]_{,j} + f_i = 0 \tag{45}$$

Further, to apply this simplified strain gradient theory to problems that favor a displacement formulation, the displacement form of the theory is derived. Combining Eqs. (17), (29) and (45) results in displacement form of the equilibrium equation

$$\begin{aligned}&\lambda * \mathrm{d}u_{p,pk} + G * \mathrm{d}\left(u_{p,pk} + u_{k,pp}\right)\\ &- \nabla^2\left[A_1 * \mathrm{d}u_{p,pk} + A_2 * \mathrm{d}\left(u_{p,pk} + \mathrm{d}u_{k,pp}\right)\right] + f_k = 0\end{aligned} \tag{46}$$

where ∇^2 is the Laplacian operator.

There are still two extra parameters $A_1,\ A_2$ that are independent of traditional materials parameters. However, in classical strain gradient theories high-order material parameters and classical material parameters are always related. Thus, we introduced a viscoelastic gradient parameter to link these two higher-order material parameters with the classical material parameters, which is inspired by the relations between Lame constants and the gradient parameter in the simplified strain gradient elasticity theory, Eq. (3). The relations are written as

$$A_1(t) = \lambda(t) * \mathrm{d}c(t),\ A_2(t) = G(t) * \mathrm{d}c(t) \tag{47}$$

where $c(t)$ is the viscoelastic gradient parameter, which can also be called the material characteristic scale parameter. This gradient parameter is related to both strain gradient effect and viscous effect of a solid. Therefore, after substituting Eq. (47) into Eq. (45) or (46), only one extra material parameter $c(t)$ is left in our gradient viscoelasticity theory. In the next section, we will present the concrete form of $c(t)$ by building higher-order viscoelastic models.

Inserting Eq. (47) into Eq. (46) then gives

$$\begin{aligned}&\lambda * \mathrm{d}u_{p,pk} + G * \mathrm{d}\left(u_{p,pk} + u_{k,pp}\right)\\ &- \nabla^2\left[\lambda * \mathrm{d}c * \mathrm{d}u_{p,pk} + G * \mathrm{d}c * \mathrm{d}\left(u_{p,pk} + \mathrm{d}u_{k,pp}\right)\right] + f_k = 0\end{aligned} \tag{48}$$

Heretofore, all equations in our strain gradient viscoelasticity theory are represented with Stieltjes' convolution, and it is a little complex in form. Moreover, a correspondence principle between gradient viscoelasticity in Laplace phase space and gradient elasticity can be given using the equations after Laplace transformation. Hence, the Laplace transform is used in our strain gradient viscoelastic equations.

Firstly, Eqs. (16) and (44) can be converted to the Laplace transform of the Cauchy stress and viscoelastic double stress, respectively giving

$$\bar{\sigma}_{ij} = \delta_{ij} s\bar{\lambda}\bar{\varepsilon}_{pp} + 2s\bar{G}\bar{\varepsilon}_{ij} \tag{49}$$

$$\bar{\tau}_{ijk} = \delta_{ij} s\bar{A}_1\bar{\varepsilon}_{ppk} + 2s\bar{A}_2\bar{\varepsilon}_{ijk} \tag{50}$$

The Laplace transformed parameters, $\bar{A}_1$ and $\bar{A}_2$, can also be given

$$\bar{A}_1 = s\bar{c}\,\bar{\lambda},\ \bar{A}_2 = s\bar{c}\,\bar{G} \tag{51}$$

Therefore, the Laplace transform of the differential equilibrium Eq. (40) yields, together with Eqs. (49), (50) and,

$$\left[\left(\delta_{ij} s\bar{\lambda}\bar{\varepsilon}_{pp} + 2s\bar{G}\bar{\varepsilon}_{ij}\right) - s\bar{c}\left(\delta_{ij} s\bar{\lambda}\bar{\varepsilon}_{ppk} + 2s\bar{G}\bar{\varepsilon}_{ijk}\right)_{,k}\right]_{,j} + \bar{f}_i = 0 \tag{52}$$

Also, Eq. (52) can be obtained directly by applying the Laplace transform to Eq. (45). Applying the Laplace transform to Eq. (48), the displacement form of the equilibrium equation is

$$\left(1 - s\bar{c}\,\nabla^2\right)\left[s\bar{\lambda}\bar{u}_{p,pk} + 2s\bar{G}\left(\bar{u}_{p,pk} + \bar{u}_{k,pp}\right)\right] + \bar{f}_k = 0 \tag{53}$$

In conclusion, equations in our strain gradient viscoelasticity theory and the classical simplified strain gradient elasticity theory can be given in a corresponding form, as shown in Table 1. Therefore, gradient-dependent viscosity problems can be solved based on solutions of the strain gradient elastic ones. Solving problems will be much easier using this correspondence principle.

4. High-Order viscoelastic model

In order to solve the governing equilibrium equation in the strain gradient viscoelasticity theory and obtain the concrete formulation of the viscoelastic gradient parameter, the relation between double stress and strain gradient must be presented. By analogy with the classical viscoelastic model, a high-order three-parameter viscoelastic model is introduced, as shown in Fig. 2. The circle with an arrow means that the element bears moment.

In Fig. 2, ζ_1 is the higher-order dashpot. H_0 and H_1 are higher-order elastic components, which represent the higher-order modulus. The relations between higher-order modulus and traditional modulus are the same as the ones in the strain gradient elasticity theory (Eq. (3))

$$\begin{aligned}H_0 &= c_e E_0\\ H_1 &= c_e E_1\end{aligned} \tag{54}$$

where c_e is the gradient parameter in strain gradient elasticity theories without considering viscous effect, E_0 and E_1 denote the traditional modulus.

Thus, relations between the double stress and strain gradient of the higher-order viscoelastic model can be given as

$$\begin{cases}\tau = \tau_0 + \tau_1\\ \dot{\varepsilon}_{,k} = \frac{\dot{\tau}_1}{H_1} + \frac{\tau_1}{\zeta_1} = \frac{\dot{\tau}_1}{c_e E_1} + \frac{\tau_1}{\zeta_1}\\ \varepsilon_{,k} = \frac{\tau_0}{H_0} = \frac{\tau_0}{c_e E_0}\end{cases} \tag{55}$$

Applying the Laplace transform, the higher-order stress–strain relation can be obtained

$$\bar{\tau} = \left(c_e E_0 + \frac{c_e E_1 \zeta_1 s}{\zeta_1 s + c_e E_1}\right)\bar{\varepsilon}_{,k} = s\bar{A}(s)\bar{\varepsilon}_{,k} \tag{56}$$

where $A(t)$ is defined as the high order relaxation modulus of isotropic solids, $\bar{A}(s)$ is in Laplace space, and the definition is similar to Eq. (22)

$$\bar{A}(s) = c_e\left(\frac{E_0}{s} + \frac{E_1\kappa_g}{1 + \kappa_g s}\right) \tag{57}$$

Applying the inverse Laplace transform, the higher-order relaxation modulus is given

$$A(t) = c_e\left(E_0 + E_1 e^{-t/\kappa_g}\right) \tag{58}$$

where

$$\kappa_g = \frac{\zeta_1}{H_1} = \frac{\zeta_1}{c_e E_1} \tag{59}$$

where κ_g is defined as the relaxation time of higher-order viscoelastic models, corresponding to a specific relaxation time of the Maxwell unit at the nano-scale. The value of $\kappa_g$should be much smaller than τ_g of the Maxwell unit at macroscopic scale.

Using the correspondence principle, similar to Eq. (3) or (54), the relation between relaxation function and higher-order relaxation function can be obtained

$$A(t) = E(t) * \mathrm{d}c(t) \tag{60}$$

where $E(t)$ is the relaxation function, as given by Eq. (23). The Laplace transform of Eq. (60) is

Table 1
Gradient-dependent elastic quantities and their viscoelastic analogs.

Strain gradient elasticity	Strain gradient Viscoelasticity
$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$	$\bar{\sigma}_{ij} = s\bar{C}_{ijkl}\bar{\varepsilon}_{kl}$
$\sigma_{ij} = \delta_{ij}\lambda\varepsilon_{pp} + 2G\varepsilon_{ij}$	$\bar{\sigma}_{ij} = \delta_{ij}s\bar{\lambda}\bar{\varepsilon}_{pp} + 2s\bar{G}\bar{\varepsilon}_{ij}$
$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$	$\bar{\varepsilon}_{ij} = \frac{1}{2}\left(\bar{u}_{i,j} + \bar{u}_{j,i}\right)$
$\tau_{ijk} = A_{ijklmn}\varepsilon_{lmn}$	$\bar{\tau}_{ijk} = s\bar{A}_{ijklmn}\bar{\varepsilon}_{lmn}$
$\tau_{ijk} = \delta_{ij}A_1\varepsilon_{ppk} + 2A_2\varepsilon_{ijk}$	$\bar{\tau}_{ijk} = \delta_{ij}s\bar{A}_1\bar{\varepsilon}_{ppk} + 2s\bar{A}_2\bar{\varepsilon}_{ijk}$
$\varepsilon_{ijk} = \varepsilon_{ij,k} = \frac{1}{2}(u_{i,jk} + u_{j,ik})$	$\bar{\varepsilon}_{ijk} = \bar{\varepsilon}_{ij,k} = \frac{1}{2}\left(\bar{u}_{i,jk} + \bar{u}_{j,ik}\right)$
$A_1 = c\lambda,\ A_2 = cG$	$\bar{A}_1 = s\bar{c}\bar{\lambda},\ \bar{A}_2 = s\bar{c}\bar{G}$
$(\sigma_{ij} - \tau_{ijk,k})_{,j} + f_i = 0$	$\left(\bar{\sigma}_{ij} - \bar{\tau}_{ijk,k}\right)_{,j} + \bar{f}_i = 0$
$\left(1 - c\nabla^2\right)\left[\lambda u_{p,pk} + 2G(u_{p,pk} + u_{k,pp})\right] + f_k = 0$	$\left(1 - s\bar{c}\nabla^2\right)\left[s\bar{\lambda}\bar{u}_{p,pk} + 2s\bar{G}\left(\bar{u}_{p,pk} + \bar{u}_{k,pp}\right)\right] + \bar{f}_k = 0$

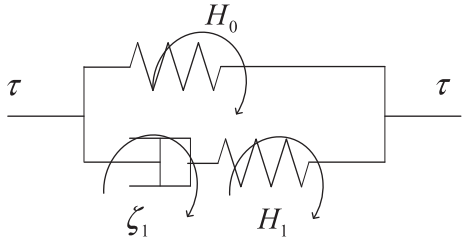

Fig. 2. The higher-order three-parameter viscoelastic model.

$$\bar{A} = s\bar{c}\,\bar{E} \tag{61}$$

Hence, the viscoelastic gradient parameter in Laplace space $\bar{c}(s)$ is

$$\bar{c} = \frac{\bar{A}}{s\bar{E}} \tag{62}$$

For the three-parameter viscoelastic models, substituting Eqs. (22) and into Eq. (57), the viscoelastic gradient parameter can be obtained

$$\bar{c} = c_e\left(\frac{E_0}{s} + \frac{E_1\kappa_g}{1 + \kappa_g s}\right)/s\left(\frac{E_0}{s} + \frac{E_1\tau_g}{1 + \tau_g s}\right) \tag{63}$$

Applying the inverse Laplace transform, the viscoelastic gradient parameter can be obtained.

$$c(t) = c_e\left\{1 + \frac{E_1(\kappa_g - \tau_g)}{E_0\kappa_g - (E_0 + E_1)\tau_g}\left[e^{\frac{-t}{\kappa_g}} - e^{\frac{-E_0 t}{(E_0+E_1)\tau_g}}\right]\right\} \tag{64}$$

This viscoelastic gradient parameter changes over time, but it has a limit value. For $t = 0$, the viscoelastic gradient parameter is

$$c_0 = c_e \tag{65}$$

For $t \to \infty$, the viscoelastic gradient parameter is

$$c_\infty \simeq c_e \tag{66}$$

c_0 and c_∞ can be called as the transient and steady-state gradient parameter, respectively.

If the classical and higher-order viscosity are neglected, which means there are no dashpot or higher-order dashpot in viscoelastic models, Eq. (64) can be reduced to

$$c = c_e \tag{67}$$

This indicates that the strain gradient viscoelasticity can be reduced to the strain gradient elasticity (Altan and Aifantis, 1997; Aifantis, 1992; Gao and Park, 2007) when the viscous elements are removed. Furthermore, the relation between $c(t)$ and c_e supplies an easier way to use our strain gradient viscoelasticity. The gradient parameter of a solid is always difficult to be determined. However, numerous studies have been done around strain gradient elasticity theories, and gradient parameters of many materials have been obtained or given the range. Therefore, the viscoelastic gradient parameter can be determined conveniently by using Eq. (64), and the more reasonable and accurate results can be obtained.

From Eq. (64) the constitutive parameter, $c(t)$, is invariant with respect to change of origin of time, $c(0) \equiv c_e$. Then the constitutive parameter varies with time due to gradient viscoelastic effect depended on the materials parameters $(c_e, E_0, E_1, \tau_g, k_g)$.

To present the relation between $c(t)$ and c_e, the distribution of the normalized gradient parameter c/c_e along the dimensionless time t/κ_g is illustrated, as shown in Fig. 3 for different values of κ_g/τ_g. The time-dependent property of this viscoelastic gradient parameter c/c_ein our viscoelasticity theory is checked as follows. The normalized gradient parameter is higher than one when $\kappa_g/\tau_g > 1$, and smaller than one when $\kappa_g/\tau_g < 1$. Referring to the research conclusions for many cross-scale problems based on the strain gradient theory, higher order effect is often predominated under the nano-/micron scale, and it is neglectable at macroscopic scale. Similarily, the high-order viscosity is related to the evolution in nano-/micro-structure, and should be more rate-sensitive than

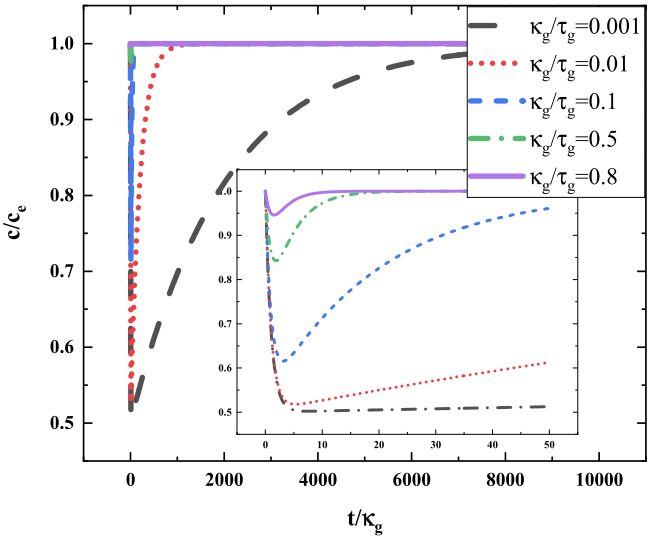

Fig. 3. Variation of the normalized gradient parameter with normalized time. The moduli E_0 and E_1 have the same value when plotting.

conventional viscosity. Thus, the high-order relaxation time κ_g should be much shorter than the classical relaxation time τ_g. And the reasonable case corresponds to $\kappa_g/\tau_g < 1$, so that we shall pay our attention to this case investigation.

For $\kappa_g/\tau_g < 1$, Fig. 3 shows variation of the normalized gradient parameter with the normalized time, t/κ_g, for several cases of κ_g/τ_gvalues, 0.001,0.01, 0.1, 0.5, 0.8. From Fig. 3, one can find that the normalized gradient parameter is always smaller than one for every case. That is to say, that the gradient parameter in strain gradient viscoelasticity theory is always smaller than that in classical strain gradient elasticity theory. Therefore, the strain gradient viscoelasticity theory can be used to describe the softening phenomenon of advanced materials at micro- or nano-scale. When the ratio of the specific relaxation times of the Maxwell unit at the nano-scale and at the macroscopic scale,κ_g/τ_g, is much smaller, as seen 0.001, the softening degree is much higher, and the ratio c/c_etends to its minimum value 0.5 within a time region corresponding to $\kappa_g/\tau_g \to 0$.

When $\kappa_g/\tau_g \to 1$, the normalized gradient parameter stays as one along time, referring to Eq. (64), and it can also be proved briefly. Firstly, $\kappa_g/\tau_g = 1$ leads to

$$A(t) = c_e\left(E_0 + E_1 e^{-t/\kappa_g}\right) = c_e\left(E_0 + E_1 e^{-t/\tau_g}\right) \tag{68}$$

Then, Eq. (62) becomes

$$\bar{c} = \frac{\bar{A}}{s\bar{E}} = \frac{c_e}{s} \tag{69}$$

Thus, Eq. (69) leads to $c = c_e$. It can be understood that the viscoelastic gradient parameter in our definition no longer shows time-dependent property when $\kappa_g/\tau_g = 1$. However, it does not mean that our strain gradient viscoelasticity theory no longer exhibits viscosity. The relaxation function and high-order relaxation function are still time-dependent, as described by Eqs. (23) and (58). Maybe we can find a better way to define this viscoelastic gradient parameter to remove this singularity in the future.

In the preceding discussion of the viscoelastic gradient parameter, both classical and higher-order viscosity are considered. However, the effect of classical and higher-order viscosity should be very different. Classical viscosity should be dominant for a longer time process corresponding to a macroscopic representative cell of materials, while the higher-order viscosity should be dominant for a shorter time process corresponding to a nano-scale representative cell of materials. If the classical viscosity is neglected (set $\tau_g \to \infty$), the classical relaxation function $E(t)$ reduces to

$$E|_{\tau_g \to \infty} \simeq E_0 + E_1 \tag{70}$$

Thus, the viscoelastic gradient parameter reduces to

$$c(t) \simeq c_e \frac{E_0 + E_1 e^{-t/\kappa_g}}{E_0 + E_1} \text{ for } \tau_g \to \infty \tag{71}$$

In this consideration, $c(t)$ is a monotonically decreasing function of time, and the steady-state value is less than the gradient parameter c_e, as

$$c_\infty = c_e \frac{E_0}{E_0 + E_1} \text{ for } \tau_g \to \infty \tag{72}$$

In another case, the higher-order viscosity is neglected (set $\kappa_g \to \infty$), which means only classical viscosity comes in, the higher-order relaxation function $A(t)$ becomes

$$A|_{\kappa_g \to \infty} \simeq H_0 + H_1 = c_e(E_0 + E_1) \tag{73}$$

And the viscoelastic parameter reduces to

$$c(t) \simeq c_e \frac{E_0 + E_1}{E_0}\left(1 - e^{\frac{-E_0 t}{(E_0+E_1)\tau_g}}\right) \text{ for } \kappa_g \to \infty \tag{74}$$

Now $c(t)$ is a monotonically increasing function of time, and the steady-state value is larger than the gradient parameter c_e, as

$$c_\infty = c_e \frac{E_0 + E_1}{E_0} \text{ for } \kappa_g \to \infty \tag{75}$$

The characterization of viscosity in different cases can be archived by adjusting the value of viscous elements. As an example, the influence of viscosity for beam bending stiffness will be discussed in the next section. Besides, the material parameter $c(t)$ is correlative with the viscoelastic and higher-order viscoelastic models. $c(t)$ in Eq. (64) is just for the classical and higher-order three-parameter viscoelastic models. However, the solution corresponding to other viscoelastic models can be obtained through the same process.

5. An example: bending of beam

The strain gradient viscoelasticity theory derived above offers an opportunity to describe the contributions of both strain gradient and viscosity in micro-/nano-scale experiments. Papargyri-Beskou et al. (Papargyri-Beskou et al., 2003) developed a size-dependent Bernoulli-Euler beam model with the uniaxial stress state assumption, which means only axial non-zero strain and strain gradient are taken into account. The six-order governing equilibrium equation and boundary conditions are obtained based on the variational approach, but this model only considers the strain gradient effect. The development of strain gradient viscoelasticity case below is an extension of that given by Papargyri-Beskou. The gradient-dependent property and viscosity during the beam bending are discussed using our strain gradient viscoelasticity theory.

5.1. Gradient viscoelastic beam model

A rectangular Bernoulli-Euler beam of length L, width b, and height h is considered. The x axis coincides with the centerline of the beam, and y,$z$axes are within the cross-section, as shown by Fig. 4. Coordinate variables x, y, z are enumerated as 1, 2, 3. The displacement components are similar to the ones in the classical elasticity

$$u_1 = -z\psi(x,t),\ u_2 = 0,\ u_3 = w(x,t) \tag{76}$$

where $\psi \approx \mathrm{d}w/\mathrm{d}x = w'$ is the rotation angle of the beam cross-section for small deformation.

From Eqs. (17) and (28), the nonzero strain and strain gradient are

$$\varepsilon_{11} = -zw'' \varepsilon_{111} = -zw''',\ \varepsilon_{113} = -w'' \tag{77}$$

And from Eq. (31), the first term of variational function $\prod$ is

$$\prod_1 = \frac{1}{2}\int_V [\sigma_{11} * \mathrm{d}\varepsilon_{11} + \tau_{111} * \mathrm{d}\varepsilon_{111} + \tau_{113} * \mathrm{d}\varepsilon_{113}]\mathrm{d}V \tag{78}$$

where stresses σ_{11}, τ_{111} and τ_{113} can be obtained from Eqs. (16) and (44) , with neglecting the Poisson effect as the classical beam theories ($\lambda(t) = 0,\ 2G(t) = E(t)$ and $A_1(t) = 0,\ 2A_2(t) = A(t)$)

$$\sigma_{11} = E * \mathrm{d}\varepsilon_{11} = -zE * \mathrm{d}w'' \tag{79}$$

$$\begin{aligned} \tau_{111} &= A * \mathrm{d}\varepsilon_{111} = -zA * \mathrm{d}w''' \\ \tau_{113} &= A * \mathrm{d}\varepsilon_{113} = -A * \mathrm{d}w'' \end{aligned} \tag{80}$$

Using Eqs. (76)-(79), the first term of variational function can be found

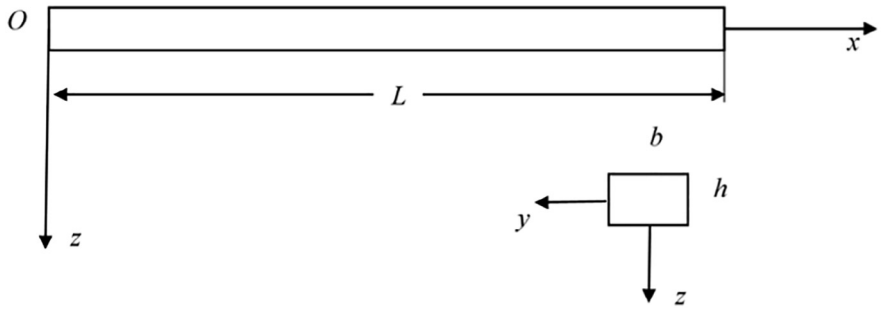

Fig. 4. Geometry and coordinate system of a beam.

$$\prod_1 = \frac{1}{2}\int_V \left[z^2 E * \mathrm{d}w'' * \mathrm{d}w'' + z^2 A * \mathrm{d}w''' * \mathrm{d}w''' + A * \mathrm{d}w'' * \mathrm{d}w''\right]\mathrm{d}V$$
$$= \frac{1}{2}\int_L \left[IE * \mathrm{d}w'' * \mathrm{d}w'' + IA * \mathrm{d}w''' * \mathrm{d}w''' + aA * \mathrm{d}w'' * \mathrm{d}w''\right]\mathrm{d}x \tag{81}$$

where $I = \int_s z^2 \mathrm{d}s$ denotes the moment of inertia, a is the area of cross-section. The resultant moment and high-order moment in our viscoelasticity may be further defined as

$$M = M(t) = (IE + aA) * \mathrm{d}w'' \tag{82}$$

$$M_h = M_h(t) = IA * \mathrm{d}w''' \tag{83}$$

Using partial integration, the first variation of $\prod_1$ is

$$\delta\prod_1 = M * \mathrm{d}(\delta w')\big|_0^L - M' * \mathrm{d}(\delta w)\big|_0^L + \int_L \left[(M'' - M'''_h) * \mathrm{d}(\delta w)\right]\mathrm{d}x + M_h * \mathrm{d}(\delta w'')\big|_0^L - M'_h * \mathrm{d}(\delta w')\big|_0^L + M''_h * \mathrm{d}(\delta w)\big|_0^L \tag{84}$$

The rest terms of variational function are written as

$$\prod_2 = -\int_L (q * \mathrm{d}w)\mathrm{d}x - (Q_s * \mathrm{d}w)\big|_0^L + (M_s * \mathrm{d}w')\big|_0^L + (M_{hs} * \mathrm{d}w'')\big|_0^L \tag{85}$$

where $q(t)$ is the external distributed force, $Q_s(t)$ is the boundary shear force, $M_s(t)$ and $M_{hs}(t)$ are the boundary moment and high-order moment. And the first variation of $\prod_2$ is

$$\delta\prod_2 = -\int_L [q * \mathrm{d}(\delta w)]\mathrm{d}x - [Q_s * \mathrm{d}(\delta w)]\big|_0^L + [M_s * \mathrm{d}(\delta w')]\big|_0^L + [M_{hs} * \mathrm{d}(\delta w'')]\big|_0^L \tag{86}$$

Thus, the first variation of variational function $\prod$ is

$$\delta\prod = \delta\left(\prod_1 + \prod_2\right) \tag{87}$$

Let $\delta\prod = 0$, and after some calculations,

$$\delta\prod = -(M' - M''_h + Q_s) * \mathrm{d}(\delta w)\Big|_0^L + (M - M'_h + M_s) * \mathrm{d}(\delta w')\big|_0^L + (M_h + M_{hs}) * \mathrm{d}(\delta w'')\big|_0^L + \int_0^L \left[(M'' - M'''_h - q) * \mathrm{d}(\delta w)\right]\mathrm{d}x = 0 \tag{88}$$

Thus, the governing equilibrium equation can be obtained

$$M'' - M'''_h = q \tag{89}$$

together with the boundary conditions at $x = 0, L$

$$\begin{aligned} M''_h - M' + Q_s &= 0 \text{ or } w = w_0 \\ M - M'_h + M_s &= 0 \text{ or } w' = w'_0 \\ M_h + M_{hs} &= 0 \text{ or } w'' = w''_0 \end{aligned} \tag{90}$$

Combining Eqs. (82), (83) and (89), the governing equilibrium equation can be written in another form

$$(EI + Aa) * \mathrm{d}w^{\mathrm{IV}} - IA * \mathrm{d}w^{\mathrm{VI}} = q \tag{91}$$

This beam bending theory is an extension of classical gradient elasticity theories. Eq. (91) includes the effect of the deformation history by using the Stieltjes integral. Hence, the viscosity exhibited during beam bending can be explained.

It should be noted that the governing equilibrium equation can also be obtained directly using our correspondence principle summarized in Section 3. The equilibrium equation in the classical strain gradient elasticity theory is (Lurie and Solyaev, 2018)

$$(EI + Aa)\mathrm{d}w^{\mathrm{IV}} - IA\mathrm{d}w^{\mathrm{VI}} = q \tag{92}$$

To obtain the corresponding equation in our strain gradient viscoelasticity theory, replace E and A with $s\bar{E}$ and $s\bar{A}$ in the Laplace space, respectively

$$\left(s\bar{E}I + s\bar{A}a\right)\bar{w}^{\mathrm{IV}} - sI\bar{A}\bar{w}^{\mathrm{VI}} = \bar{q} \tag{93}$$

The inverse Laplace transform of Eq. (93) is the solution of the gradient viscoelastic beam, that is, Eq. (91).

5.2. Bending solutions of gradient viscoelastic beam

For beam pure bending, beam size and coordinates are set as in Section 5.1. The displacement components can be written as

$$u_1 = \rho xz,\ \mathrm{u}_2 = -\rho\nu yz,\ \mathrm{u}_3 = \frac{1}{2}\rho\nu\left(y^2 - z^2\right) - \frac{1}{2}\rho x^2 \tag{94}$$

where ν is the Passion ratio, $\rho = \rho(t)$ is the curvature of the central axis of the beam after bending in the yz-plane, and it is the function of time in our strain gradient viscoelasticity theory. The components of strain and strain gradient are

$$\begin{aligned} &\varepsilon_{11} = \rho z,\ \varepsilon_{22} = -\nu\rho z,\ \varepsilon_{33} = -\nu\rho z \\ &\varepsilon_{113} = \rho,\ \varepsilon_{223} = -\nu\rho,\ \varepsilon_{333} = -\nu\rho \end{aligned} \tag{95}$$

Setting the bending moment $M = M(t)$, the functional $\prod(t)$ can be written as

$$\prod = \int_V \left[\frac{1}{2}z^2 E * \mathrm{d}\rho * \mathrm{d}\rho + \frac{1}{2}A * \mathrm{d}\rho * \mathrm{d}\rho\right]\mathrm{d}V - LM * \mathrm{d}\rho \tag{96}$$

where $E(t)$ is the stiffness function, and $A(t)$ is the high-order stiffness function.

The resulting first variation of Eq. (96) is written as

$$\delta\prod = \int_V [z^2 E * \mathrm{d}\rho * \mathrm{d}(\delta\rho) + A * \mathrm{d}\rho * \mathrm{d}(\delta\rho)]\mathrm{d}V - LM * \mathrm{d}(\delta\rho) \tag{97}$$

Let $\delta\prod = 0$,

$$\int_V (z^2 E * \mathrm{d}\rho + A * \mathrm{d}\rho)\mathrm{d}V - LM = 0 \tag{98}$$

The Laplace transformation of Eq. (98) is

$$\int_V s\left(z^2 \bar{E}\,\bar{\rho} + \bar{A}\,\bar{\rho}\right)\mathrm{d}V - L\bar{M} = 0 \tag{99}$$

The twist angle can be obtained

$$\bar{\rho} = \frac{\bar{M}}{s\bar{E}I + s\bar{A}a} \tag{100}$$

where a denotes the area of the cross-section, I denotes the moment of inertia. Thus, the gradient-dependent viscoelastic bending stiffness $D(t)$ can be defined, and the stiffness in Laplace space is expressed as

$$s\bar{D} = s\bar{E}I + s\bar{A}a \tag{101}$$

Using the classical and high-order three-parameter viscoelastic models, the twist angle can be presented as

$$\bar{\rho} = \frac{\bar{M}}{s\bar{E}I + s\bar{A}a} = \frac{\bar{M}}{s\left[\left(\frac{E_0}{s} + \frac{E_1\tau_g}{1+\tau_g s}\right)I + a\left(\frac{c_e E_0}{s} + \frac{c_e E_1\kappa_g}{1+\kappa_g s}\right)\right]} \tag{102}$$

And the bending stiffness in the Laplace space can be updated from Eq. (102) as

$$\bar{D} = \bar{E}I + \bar{A}a = \left(\frac{E_0}{s} + \frac{E_1\tau_g}{1+\tau_g s}\right)I + ac_e\left(\frac{E_0}{s} + \frac{E_1\kappa_g}{1+\kappa_g s}\right) \tag{103}$$

Applying the inverse Laplace transform, the gradient-dependent viscoelastic bending stiffness is

$$D(t) = E_0(I + ac_e) + E_1\left(I\mathrm{e}^{-t/\tau_g} + ac_e\mathrm{e}^{-t/\kappa_g}\right) \tag{104}$$

At the initial point, that is $t = 0$, the bending stiffness is

$$D_0 = (E_0 + E_1)(I + ac_e) \tag{105}$$

It is called transient gradient elastic stiffness. When time approach infinity, that is, viscous effects are all released, the bending stiffness becomes

$$D_\infty = E_0(I + ac_e) \tag{106}$$

D_∞ is called steady-state gradient elastic stiffness. The gradient viscoelastic stiffness changes over time, but always between the transient and steady-state gradient elastic stiffness.

Substituting Eq. (64), the gradient-dependent viscoelastic bending stiffness is

$$D(t) = E_0 I + E_1 I\mathrm{e}^{-\frac{t}{\tau_g}} + ac(t)\left(E_0 + E_1\mathrm{e}^{-\frac{t}{\kappa_g}}\right)\Big/\left[1 + \frac{E_1\kappa_g(\kappa_g - \tau_g)}{E_0\kappa_g - (E_0+E_1)\tau_g}\left[\exp\left(-\frac{t}{\kappa_g}\right) - \exp\left(\frac{-E_0 t}{(E_0+E_1)\tau_g}\right)\right]\right] \tag{107}$$

Similarly, by using the correspondence principle summarized in Section 3, Eq. (101) can also be obtained directly without complicated derivations. The corresponding gradient-dependent elastic bending stiffness is (Zhou et al., 2016)

$$D_e = EI + Aa \tag{108}$$

Then, replacing E, A and D_e with $s\bar{E}$, $s\bar{A}$ and $s\bar{D}$ in the Laplace space, respectively, the Laplace transformation of corresponding stiffness in our strain gradient viscoelasticity theory can be obtained, as Eq. (101).

There is a relationship between $c(t)$ and c_e, as in Eq. (64). Thus, c_e is discussed as a variable of the bending stiffness here, as presented by Eq. (104). In this way, the viscosity can also be separated from the viscoelastic gradient parameter. For the characterization of objective laws, our gradient viscoelastic bending stiffness $D(t)$ is nondimensionalized by the steady-state gradient elastic stiffness D_∞, and time is nondimensionalized by the relaxation time κ_g. For illustration, parameters of the beam considered here are taken as: $b = h = 1$, $E_0 = E_1$. Hence, the transient gradient elastic stiffness D_0 (for $t = 0$) is equal to twice the steady-state one D_∞ (for $t \to \infty$), and the dimensionless stiffness equal to 2 at $t = 0$. The distributions of the dimensionless bending stiffness over dimensionless time are displayed in Fig. 5 for different ratios of gradient parameter and the square of beam height. The dimensionless bending stiffness will approach unity as the time approaches infinity (Eq. (104)), which means the results obtained by our theory tend to the one of steady-state gradient elasticity. This is a general law in viscoelasticity. Besides, Fig. 5 shows different gradient parameters make the gradient-dependent viscoelastic solutions tend to the steady-state gradient elastic ones at different rates. For larger h^2/c_e, thick beam case, the dimensionless stiffness enters a more table stage earlier, but it is still gradually tending to the steady-state value.

Then, the influence of the ratio between κ_g and τ_g on the bending stiffness is investigated. The dependence of dimensionless bending stiffness upon t/κ_g is given in Fig. 6 for selected values of κ_g/τ_g. In this illustration, κ_g takes a fixed value, and τ_g are different multiple of κ_g. The trends of the dimensionless bending stiffness have a qualitative difference when κ_g/τ_g is different. However, the dimensionless bending stiffness still approaches unity as the time approaches infinity, which means the gradient

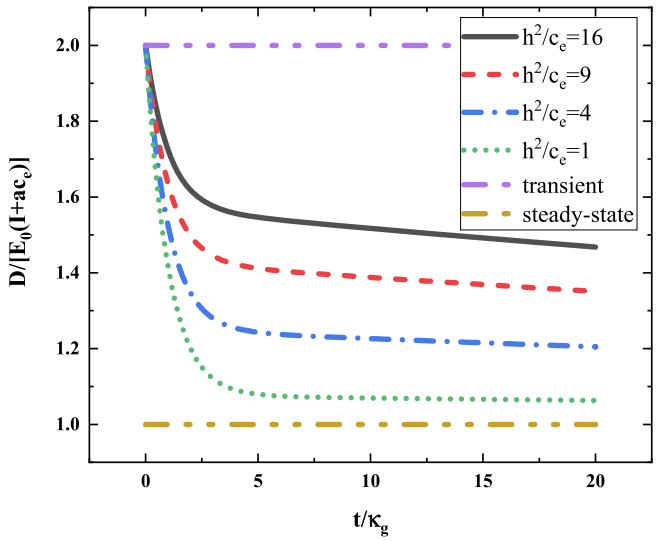

Fig. 5. Dimensionless bending stiffness as the function of time for different gradient parameters. Setting $\kappa_g/\tau_g = 0.01$. The assumption $E_0 = E_1$ brings the result that the dimensionless stiffness is 2 when $t/\kappa_g = 0$.

viscoelastic stiffness is close to the steady-state gradient elastic stiffness. Fig. 6 shows the different rates of change in dimensionless bending stiffness for different κ_g/τ_g. For smaller κ_g/τ_g, the downward trend of the dimensionless stiffness is slower, which means that it will take longer to the steady-state. Besides, when the gradient parameter is larger (Fig. 6b), the gradient viscoelastic stiffness approaches the steady-state gradient elastic stiffness faster under the same κ_g/τ_g.

Differences between elasticity and our gradient viscoelasticity are discussed, as shown in Fig. 7. The bending stiffness is nondimensionalized by the steady-state elastic stiffness. Steady-state elasticity is a reference, and it corresponds to $t \to \infty, c_e = 0$, that is, $D(t) = D_{se} = E_0 I$; thus, the dimensionless stiffness is 1. Transient elasticity is another reference, and it corresponds to $t \to 0, c_e = 0$, that is, $D(t) = D_{se} = (E_0 + E_1)I$; thus, the dimensionless stiffness is 2 under $E_0 = E_1$. At $t = 0$, our gradient viscoelasticity is the same as the transient gradient elasticity. The bending stiffness tends to the transient elastic stiffness as h^2/c_e approaches infinity. At $t \to \infty$, our gradient viscoelasticity tends to the steady-state gradient elasticity, and the bending stiffness tends to the steady-state elastic stiffness as h^2/c_e approaches infinity. However, for time

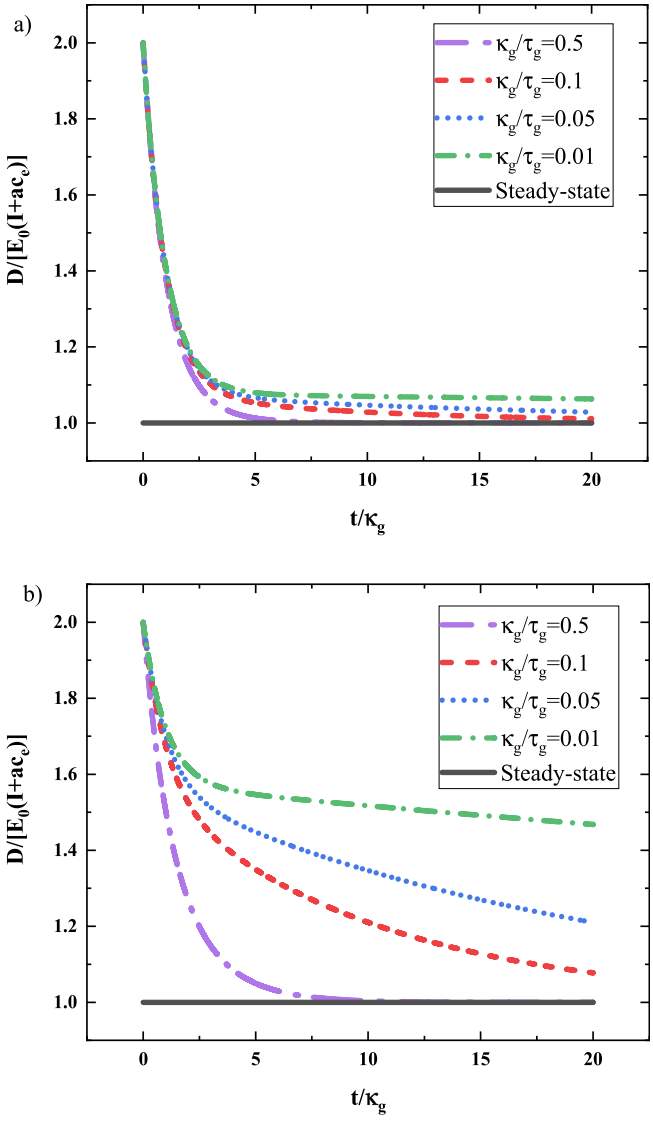

Fig. 6. Dimensionless bending stiffness as the function of dimensionless time for different ratios of κ_g and τ_g. The same assumption $E_0 = E_1$ is used. a) thin beam case, $h^2/c_e = 1$. b) thick beam case, $h^2/c_e = 16$.

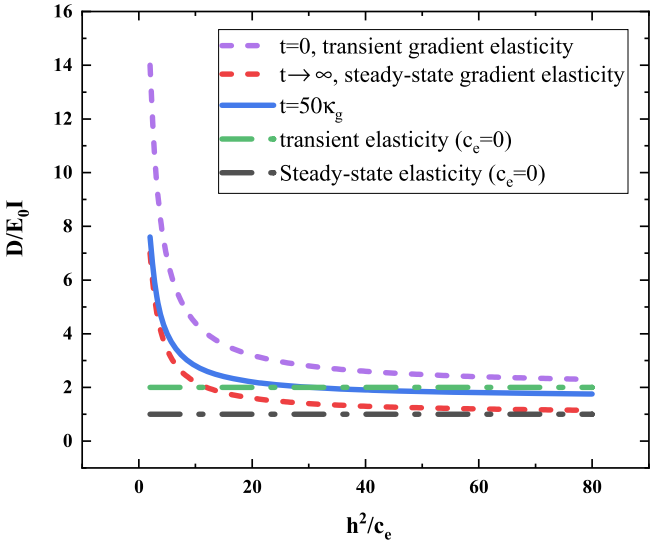

Fig. 7. Dimensionless bending stiffness as the function of h^2/c_e for different moments. The same assumption $E_0 = E_1$ is used, and $\kappa_g/\tau_g = 0.01$. The gradient-dependent viscoelastic bending stiffness is nondimensionalized by the steady-state elastic stiffness.

does not take these two points, values of bending stiffness always between transient and steady-state gradient elastic stiffness. Besides, solutions of transient gradient elasticity and steady-state gradient elasticity converge to the ones of transient elasticity and steady-state elasticity, respectively. This law is the same as the one given in the studies of strain elasticity theories.

In all the above discussions, both classical and high-order viscosity are included. However, as mentioned in section 4, the effect of classical and high-order viscosity may be very different. For this beam pure bending example, our gradient viscoelasticity can characterize the situation considering only classical viscosity or only high-order viscosity. If the classical viscosity is neglected (set $\tau_g \to \infty$), the bending stiffness becomes $D(t) \approx E_0(I + ac_e) + E_1(I + ac_e e^{-t/\kappa_g})$. The transient bending stiffness remains the same, but the steady-state bending stiffness becomes $D_\infty = E_0(I + ac_e) + E_1 I$. Thus, only the gradient-dependent part is time-dependent. This situation may be suitable for macroscopic non-viscous solids whose classical viscosity is always neglected. Thus, the time-dependent property exhibited by these solids at the micro-/nano-scale may be characterized by higher-order viscosity. While for macroscopic viscous solids, such as polymers, gels, time-dependent effects are obvious at both the macroscopic scale and micro-/nano-scale. If higher-order viscosity is neglected, the bending stiffness in the beam pure bending example is $D(t) \approx E_0(I + ac_e) + E_1(Ie^{-t/\tau_g} + ac_e)$, and the steady-state bending stiffness becomes $D_\infty = E_0(I + ac_e) + E_1 ac_e$. The results are illustrated in Fig. 8. It is clearly observed that the gradient-dependent viscoelastic bending stiffness approach different values when either classical or high-order viscosity is ignored, and these steady-state values are larger than the one when both classical and high-order viscosity are considered.

Through the analysis of the beam pure bending, it can be found that our strain gradient viscoelasticity theory can describe many phenomena that classical gradient elasticity theories can not describe. The simplified strain gradient elasticity theory does not contain viscosity, and it can be seen as a special case of our strain gradient viscoelasticity theory. After ignoring viscosity, our theory reduces to the gradient elasticity. Also, the gradient-dependent viscoelastic bending stiffness is related to the classical and high-order viscoelastic models. We have only used the standard three-

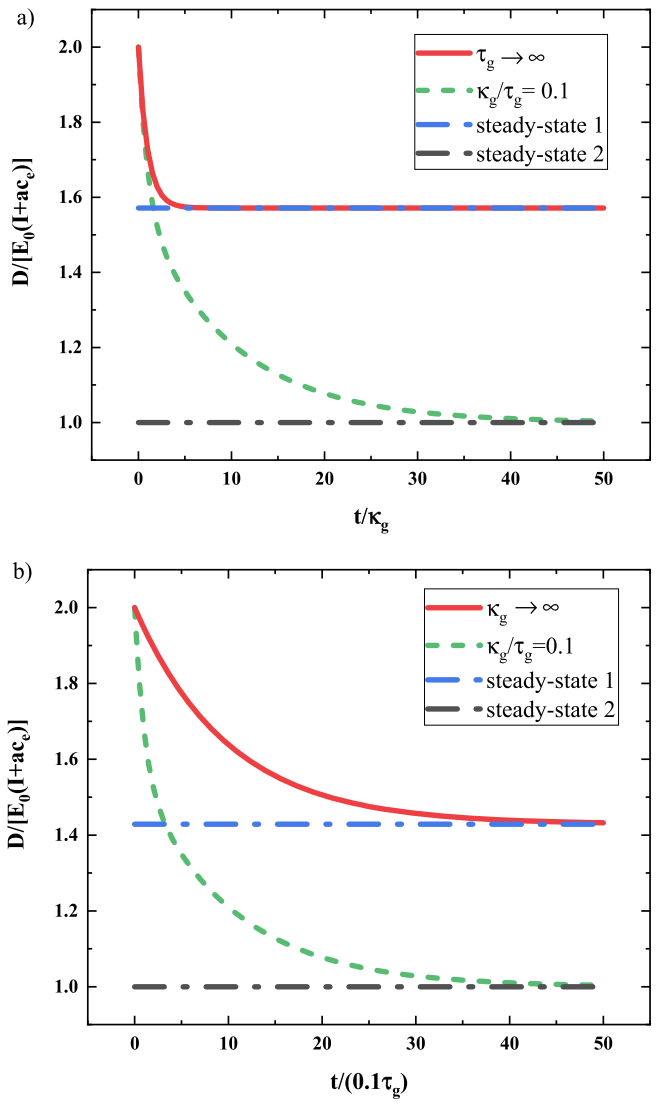

Fig. 8. Dimensionless bending stiffness as a function of time after classical (or higher-order) viscosity is ignored. The same assumption $E_0 = E_1$ is used, and $h^2/c_e = 16$. The green dash curve, $\kappa_g/\tau_g = 0.1$, is for comparison. Steady-state1 is the steady-state gradient elastic bending stiffness with $\tau_g \to \infty$ (or $\kappa_g \to \infty$). Steady-state2 is the steady-state gradient elastic bending stiffness with the normal classical viscosity.

parameter viscoelastic models for discussion. However, the results of other viscoelastic models can be obtained through a similar process.

6. Conclusion

In this paper, a strain gradient viscoelastic theory has proposed. By using this theory, the mechanical behavior of quasi-brittle advanced materials can be characterized, and both the viscous effect and strain gradient effect can be described at the micro-/nano-scale. If the strain gradient effect and viscous effect are neglected, the strain gradient viscoelastic theory can be simplified to the classical strain gradient elastic theory and the classical viscoelastic theory, respectively. In the process of deriving the theory of strain gradient viscoelasticity, the governing equilibrium equation and boundary conditions of strain gradient viscoelasticity have been obtained by using the variational principle. On this basis, the correspondence principle between the form of the strain gradient viscoelastic theory in the Laplace phase space and the strain gradient elastic theory has been derived. This correspondence principle can be used to solve the problem of strain gradient viscoelasticity. With the help of the high-order viscoelastic model, the time curve of material characteristic scale parameters in viscoelastic deformation has been obtained, which provides a way for the observation of microstructure evolution of advanced materials.

As an application example of the strain gradient viscoelastic theory, this paper has analyzed the cross-scale bending problem of the quasi-brittle advanced material beam in detail, taking into account the viscous effect and the strain gradient effect. The governing equilibrium equation and boundary conditions of the Bernoulli-Euler beam with strain gradient viscoelastic material have been established. The bending stiffness of a beam has been obtained, and the influence of viscous effect on the bending stiffness has been analyzed. The results have shown that the bending stiffness tends to be a steady-state gradient elastic stiffness with the increase of time. When the viscous effect is neglected, the bending stiffness tends to the theoretical solution of strain gradient elasticity.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Corrigendum to "A strain gradient linear viscoelasticity theory" [Int. J. Solids. Struct., 203 (2020), 197–209]

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The authors regret the typographical errors of some formulas in the article. Eight corrections are listed below:

Page	Should read
199, Eq. (3)	$A_1 = c_e\lambda,$ $A_2 = c_eG.$
199, Eq. (5)	$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} = \delta_{ij}\lambda\varepsilon_{pp} + 2G\varepsilon_{ij} = \sigma_{ji},$ $\tau_{ijk} = \frac{\partial w}{\partial \varepsilon_{ijk}} = c_e\left(\delta_{ij}\lambda\varepsilon_{ppk} + 2G\varepsilon_{ijk}\right) = \tau_{jik}.$
199, Eq. (10)	$(\sigma_{ij} - \tau_{ijk,k})n_j - D_j(n_k\tau_{ijk}) + (D_ln_l)n_kn_j\tau_{ijk} = \bar{p}_i,$ or $u_i = \bar{u}_i;$ $\tau_{ijk}n_jn_k = \bar{q}_i,$ or $u_{i,l}n_l = \frac{\partial \bar{u}_i}{\partial n}.$
199, Eq. (15)	$s_{ij}(t) = 2G(t) * \mathrm{d}e_{ij}(t),$ $\sigma_{pp}(t) = 3K(t) * \mathrm{d}\varepsilon_{pp}(t).$
200, Eq. (18)	$\bar{s}_{ij} = 2s\bar{G}\bar{e}_{ij},$ $\bar{\sigma}_{pp} = 3s\bar{K}\bar{\varepsilon}_{pp},$ $\bar{\sigma}_{ij} = 2s\bar{G}\bar{\varepsilon}_{ij} + \delta_{ij}s\bar{\lambda}\bar{\varepsilon}_{pp}.$
200, Eq. (19)	$\sigma_e = \sigma_0 + \sigma_1,$ $\dot{\varepsilon}_e = \frac{\dot{\sigma}_1}{E_1} + \frac{\sigma_1}{\eta_1},$ $\varepsilon_e = \frac{\sigma_0}{E_0}.$
201, Eq. (43)	$\sigma_{ij,j} + f_i = 0;$ $\sigma_{ij}n_j = p_i,$ or $u_i = \hat{u}_i.$
204, Eq. (77)	$\varepsilon_{11} = -zw'';$ $\varepsilon_{111} = -zw''',$ $\varepsilon_{113} = -w''.$

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